A queueing system with on-demand servers: local stability of fluid limits

Lam M. Nguyen, Alexander L. Stolyar

(Lehigh University)

INFORMS Annual Meeting
November 15, 2016

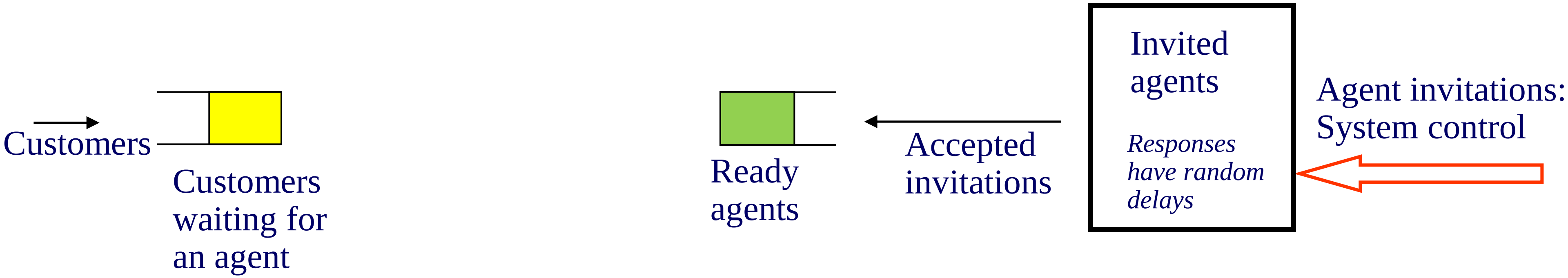
Outline

- Motivation, model and algorithm
- Fluid scale analysis
- Main result (sufficient local stability conditions)
- Numerical and simulation examples
- Discussion and future work

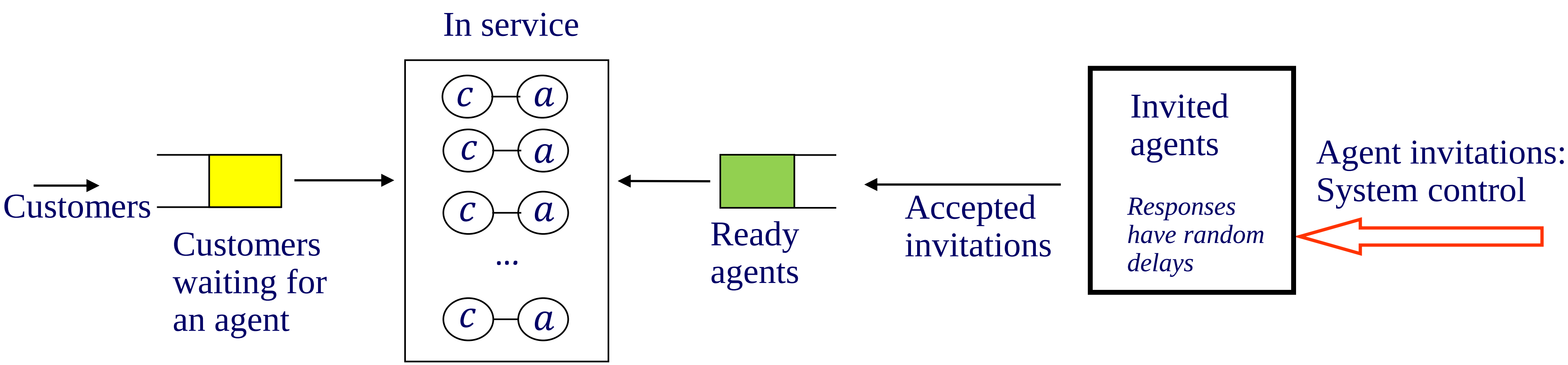
Motivation: Call center

Motivation: Call center

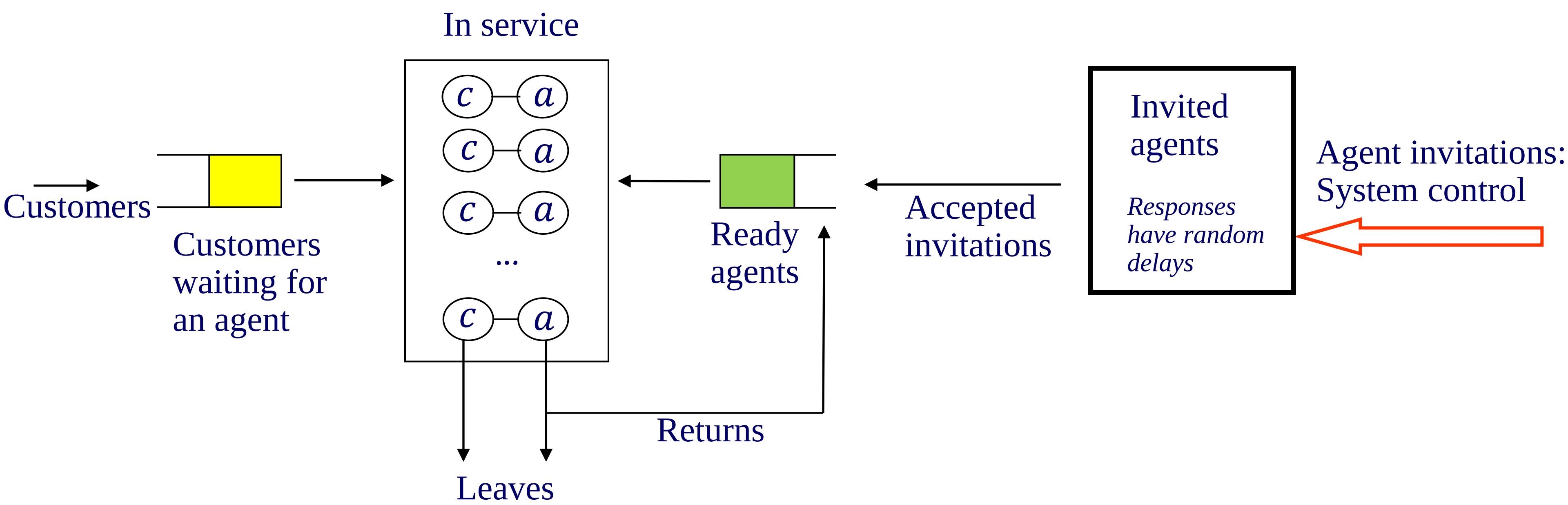
Motivation: Call center



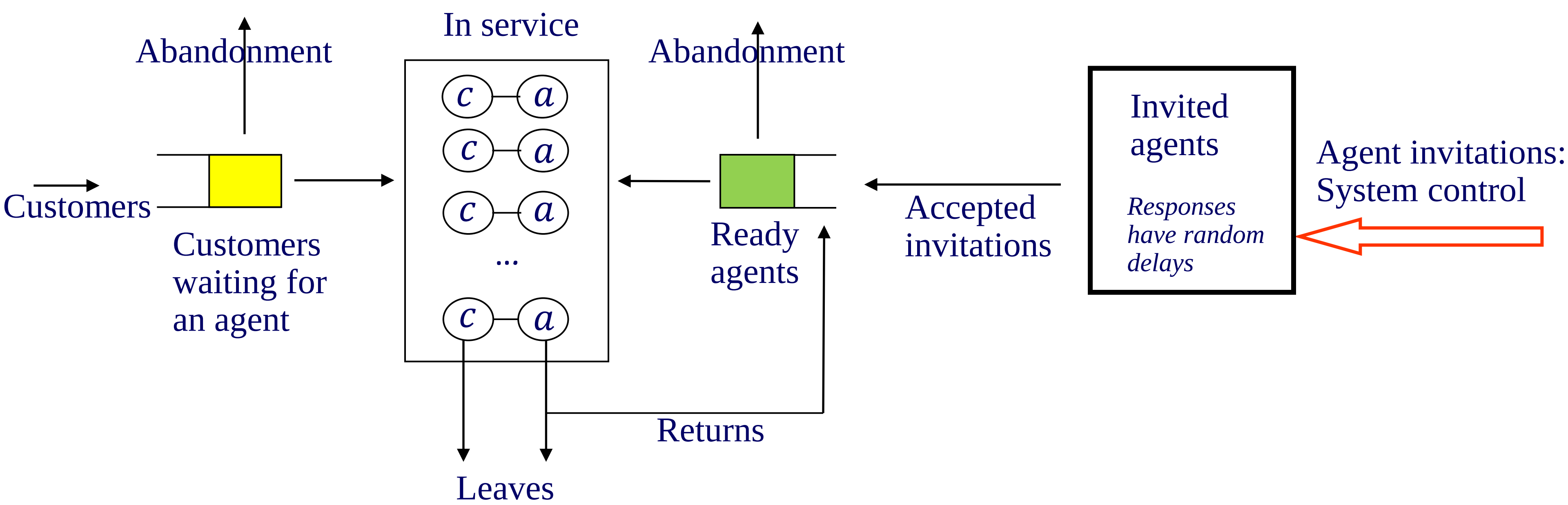
Motivation: Call center



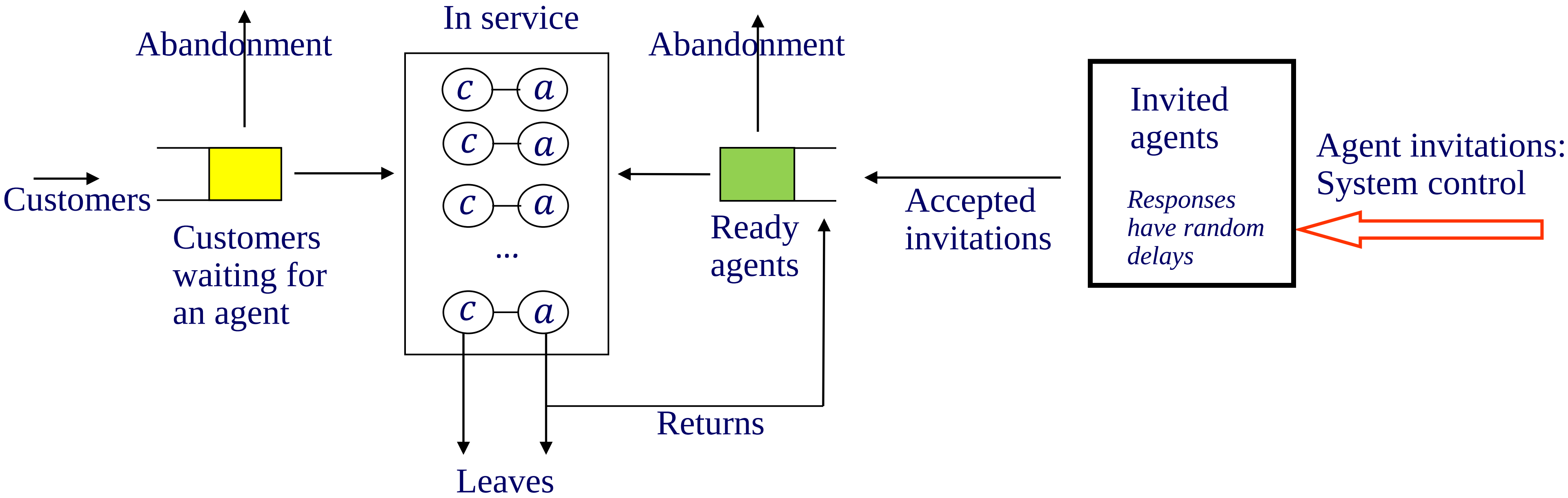
Motivation: Call center



Motivation: Call center

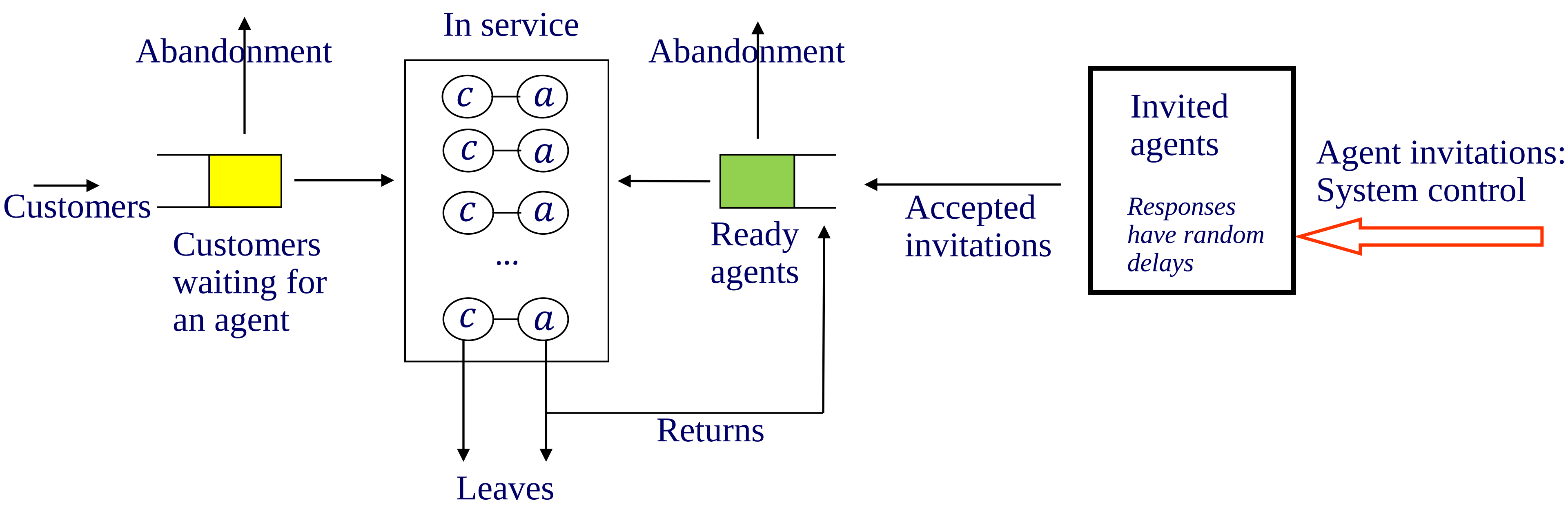


Motivation: Call center



- **Objective**: Keep delays of both customers and agents low

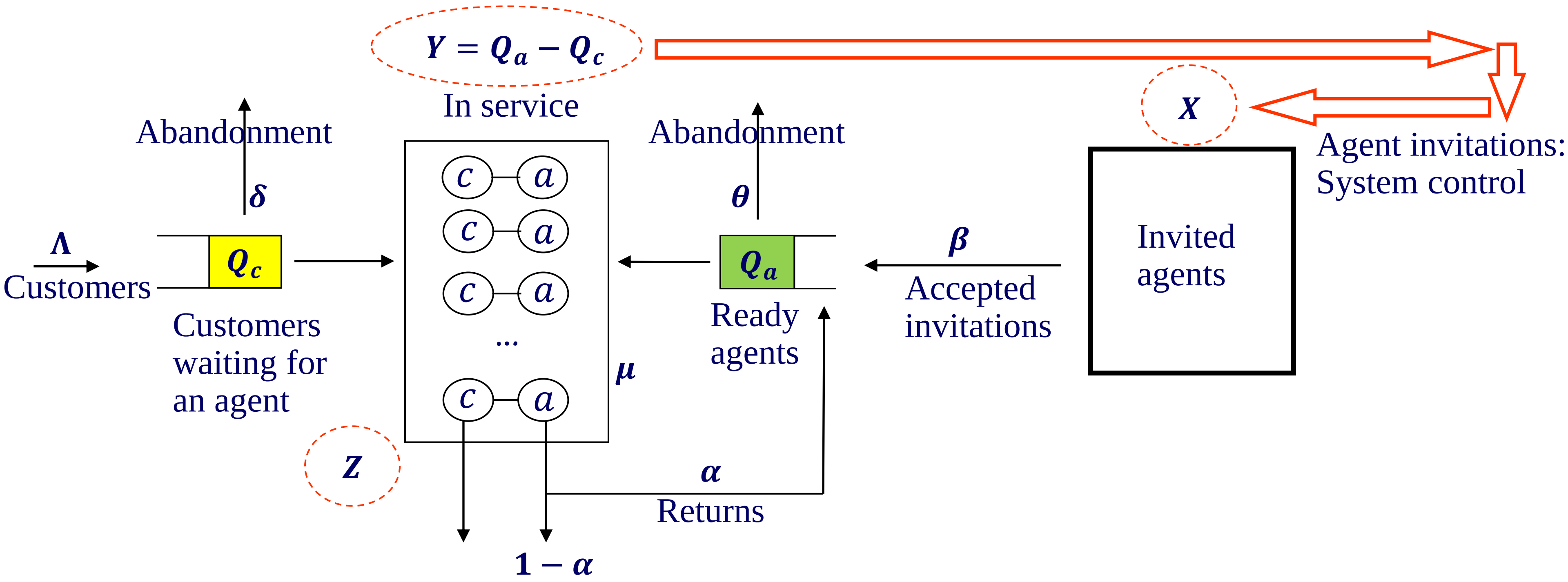
Motivation: Call center



- **Objective**: Keep delays of both customers and agents low

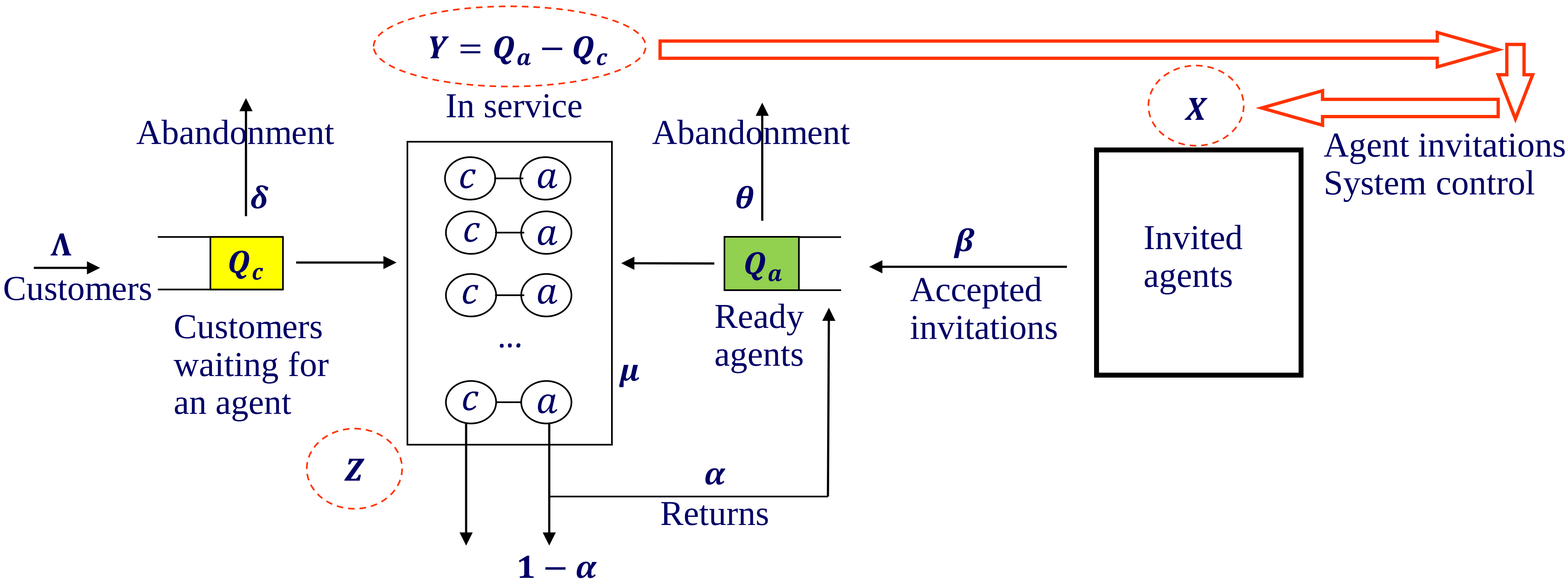
- **Other applications**:
  - Telemedicine
  - Taxi-service system
  - Assemble-to-order system

Model



$Y = Q_a - Q_c$
In service
Abandonment
δ
Abandonment
θ
X
Agent invitations:
System control
Λ
Customers
Q_c
c a
c a
c a
...
c a
Q_a
β
Invited
agents
Accepted
invitations
Customers
waiting for
an agent
Ready
agents
μ
Z
α
Returns
$1 - \alpha$

Model



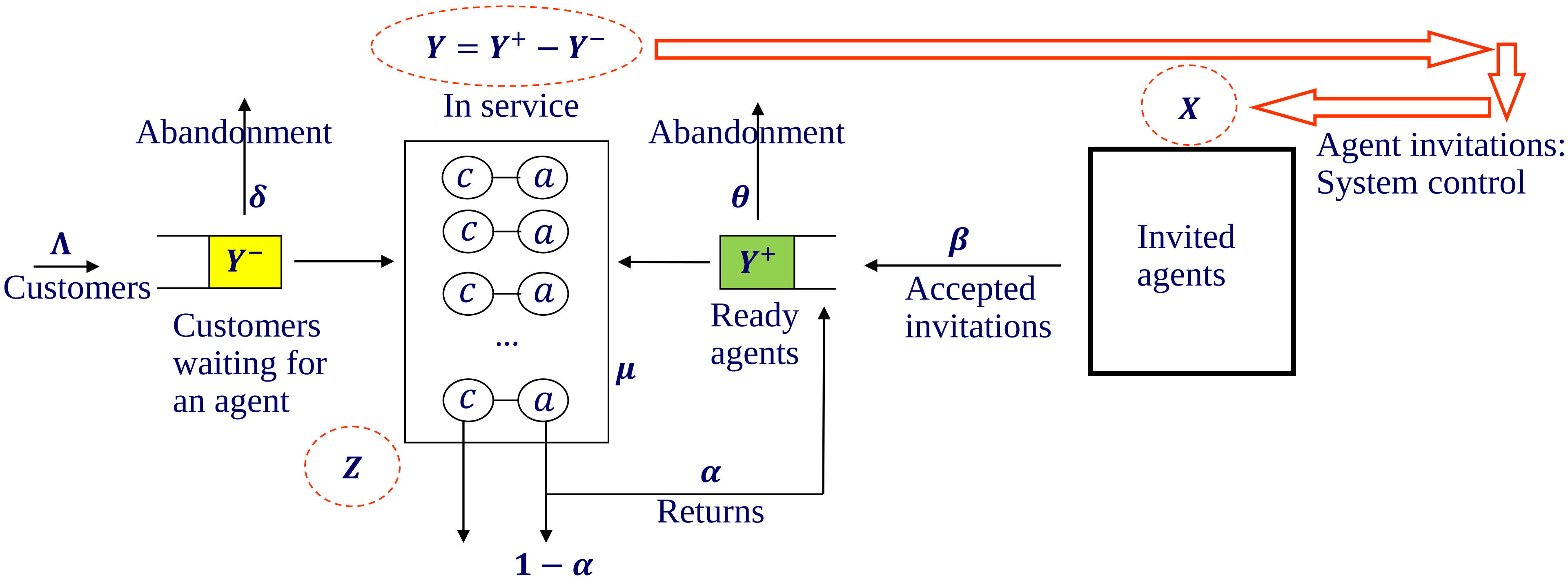
- **Non-idling condition**:
  - The head-of-the-line customer and agent are matched immediately and together go to service
  - The customer and agent queues cannot be positive simultaneously

$$Y = Y^{+} - Y^{-}$$
$$Y^{+} = max\{Y, 0\}$$
$$Y^{-} = max\{-Y, 0\}$$

$\Rightarrow$

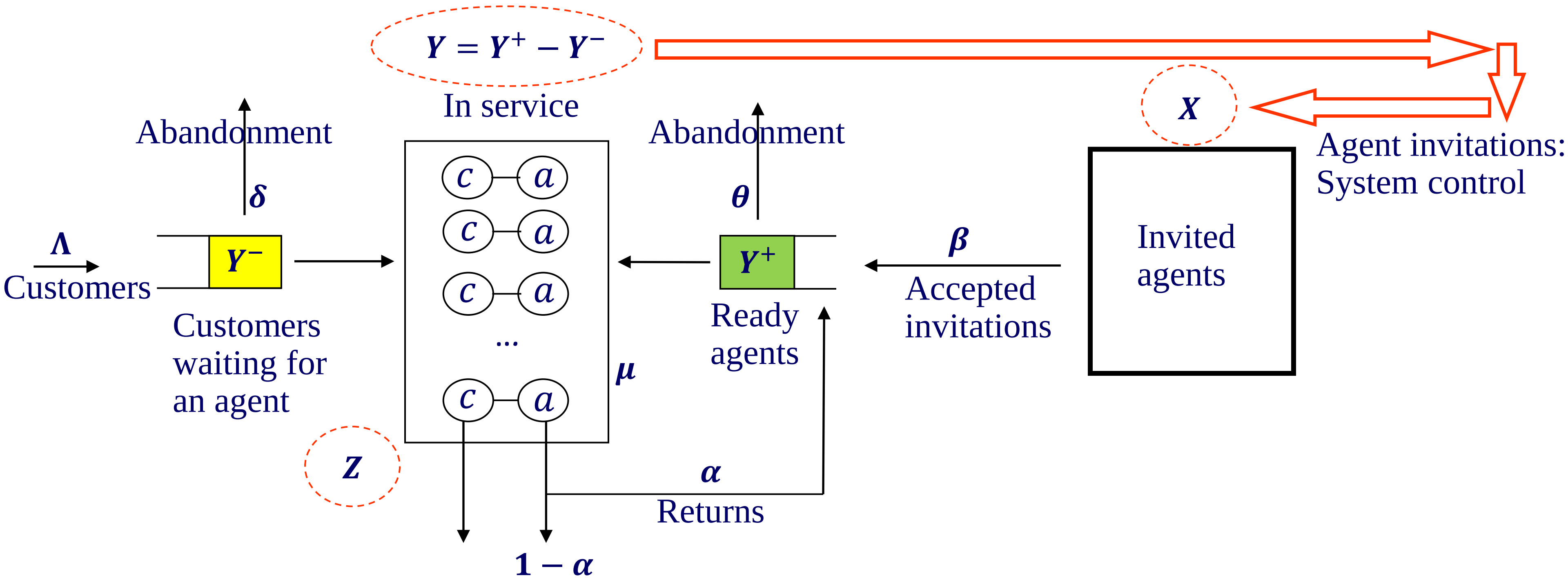
$$Q_a = Y^{+}$$
$$Q_c = Y^{-}$$

Model. Feedback algorithm



- **Feedback scheme**: [A. Stolyar et al., 2010]
  - X is incremented (on average) by $[-\gamma \Delta Y]$ each time Y changes by ΔY $(=+1$ *or* $-1)$, where $\gamma > 0$ is parameter
  - Independently, X is incremented by $-\text{sign}(Y)$ at the instantaneous rate $|\epsilon Y|$, where $\epsilon > 0$ is parameter

Model. Feedback algorithm



- **Feedback scheme**: [A. Stolyar et al., 2010]
  - X is incremented (on average) by $[-\gamma \Delta Y]$ each time Y changes by ΔY $(=+1$ *or* $-1)$, where $\gamma > 0$ is parameter
  - Independently, X is incremented by $-\text{sign}(Y)$ at the instantaneous rate $|\epsilon Y|$, where $\epsilon > 0$ is parameter

INFORMALLY:

$$(d/dt)X = -\gamma(d/dt)Y - \epsilon Y$$

$$(d/dt)Y = \beta X - \Lambda + \alpha\mu Z + \delta Y^{-} - \theta Y^{+}$$

Algorithm in detail

Algorithm parameters: $\gamma > 0$ and $\epsilon > 0$

The algorithm control the number of invited agents $X(t)$, which responds to different events during time *dt* as follows:

- **A customer arrival** with probability Λdt $\quad \Delta X(t) = \gamma$
- **An agent acceptance** with probability $\beta X(t) dt$ $\quad \Delta X(t) = -(\gamma \wedge X(t))$
- **An additional event** with probability $\epsilon |Y(t)| dt$ $\quad \Delta X(t) = -sgn\big(Y(t)\big), if\ X(t) \geq 1$
  $\Delta X(t) = 1, if\ X(t) = 0\ and\ Y(t) < 0$
  $\Delta X(t) = 0, if\ X(t) = 0\ and\ Y(t) \geq 0$
- **A service completion** with probability $\mu Z(t) dt$
  - **Agent returns to the agent queue** with probability α $\quad \Delta X(t) = -(\gamma \wedge X(t))$
  - **Agent leaves the system** with probability $1 - \alpha$ $\quad \Delta X(t) = 0$
- **A customer abandonment** with probability $\delta Y^{-}(t) dt$ $\quad \Delta X(t) = -(\gamma \wedge X(t))$
- **An agent abandonment** with probability $\theta Y^{+}(t) dt$ $\quad \Delta X(t) = \gamma$

Related work

- The model is a generalized version of

  - *Non-abandonment system* (customers and agents do not abandon their queues until they are matched, that is, $\delta = 0$ and $\theta = 0$) [L. Nguyen and A. Stolyar, 2016]

  - *Basic system* (non-abandonment system with no returning agents after service completions, that is, $\delta = 0$, $\theta = 0$, and $\alpha = 0$) [G. Pang and A. Stolyar, 2016]

- Related work

  - Double-ended queues [B. Kashyap, 1966; …]

  - Matching systems [I. Gurvich and A. Ward, 2014; …]

Process. Fluid scale analysis

- Consider the system process (X^r, Y^r, Z^r) when $\boldsymbol{r} \rightarrow \infty$, with $\Lambda = \lambda r$, while $\alpha, \beta, \mu, \delta, \theta, \epsilon, \gamma$ do not depend on r

- To match arrival rate on average: $\boldsymbol{\beta X^r + \alpha\mu Z^r - \theta(Y^r)^+ = \lambda r - \delta(Y^r)^-}$
  - X^r is $\lambda r(1-\alpha)/\beta$
  - Y^r is 0
  - Z^r is $\lambda r/\mu$

Process. Fluid scale analysis

- Consider the system process (X^r, Y^r, Z^r) when $\boldsymbol{r} \to \infty$, with $\Lambda = \lambda r$, while $\alpha, \beta, \mu, \delta, \theta, \epsilon, \gamma$ do not depend on r

- To match arrival rate on average: $\boldsymbol{\beta X^r + \alpha\mu Z^r - \theta(Y^r)^+ = \lambda r - \delta(Y^r)^-}$
  - X^r is $\lambda r(1-\alpha)/\beta$
  - Y^r is 0
  - Z^r is $\lambda r/\mu$

- Consider the process (X^r, Y^r, V^r) where $V^r = (Y^r)^+ + Z^r$

- Fluid-scaled processes with centering

$$(\bar{X}^r, \bar{Y}^r, \bar{V}^r) = r^{-1}(X^r - \lambda r(1-\alpha)/\beta, Y^r, V^r - \lambda r/\beta)$$

Fluid limit

- Fluid-scaled processes with centering

$$(\bar{X}^r, \bar{Y}^r, \bar{V}^r) = r^{-1}(X^r - \lambda r(1-\alpha)/\beta, Y^r, V^r - \lambda r/\beta)$$

- **Fluid limit**

$$(x(\cdot), y(\cdot), v(\cdot)) = \lim_{r\to\infty} (\bar{X}^r(\cdot), \bar{Y}^r(\cdot), \bar{V}^r(\cdot))$$

satisfies conditions

$$\begin{cases} x' = \begin{cases} -\gamma y' - \epsilon y, if\ x > -\dfrac{\lambda(1-\alpha)}{\beta} \\ [-\gamma y' - \epsilon y] \vee 0, if\ x = -\dfrac{\lambda(1-\alpha)}{\beta} \end{cases} \\ y' = \beta x + \alpha\mu(v - y^+) + \delta y^- - \theta y^+ \\ v' = \beta x - (1-\alpha)\mu(v - y^+) - \theta y^+ \end{cases} \qquad (1)$$

Fluid limit

- Fluid-scaled processes with centering

$$(\bar{X}^r, \bar{Y}^r, \bar{V}^r) = r^{-1}(X^r - \lambda r(1-\alpha)/\beta, Y^r, V^r - \lambda r/\beta)$$

- **Fluid limit**

$$(x(\cdot), y(\cdot), v(\cdot)) = \lim_{r\to\infty} (\bar{X}^r(\cdot), \bar{Y}^r(\cdot), \bar{V}^r(\cdot))$$

satisfies conditions

$$\begin{cases} x' = \begin{cases} -\gamma y' - \epsilon y, if\ x > -\dfrac{\lambda(1-\alpha)}{\beta} \\ [-\gamma y' - \epsilon y] \vee 0, if\ x = -\dfrac{\lambda(1-\alpha)}{\beta} \end{cases} \\ y' = \beta x + \alpha\mu(v - y^+) + \delta y^- - \theta y^+ \\ v' = \beta x - (1-\alpha)\mu(v - y^+) - \theta y^+ \end{cases} \quad (1)$$

Is the system (1) stable?

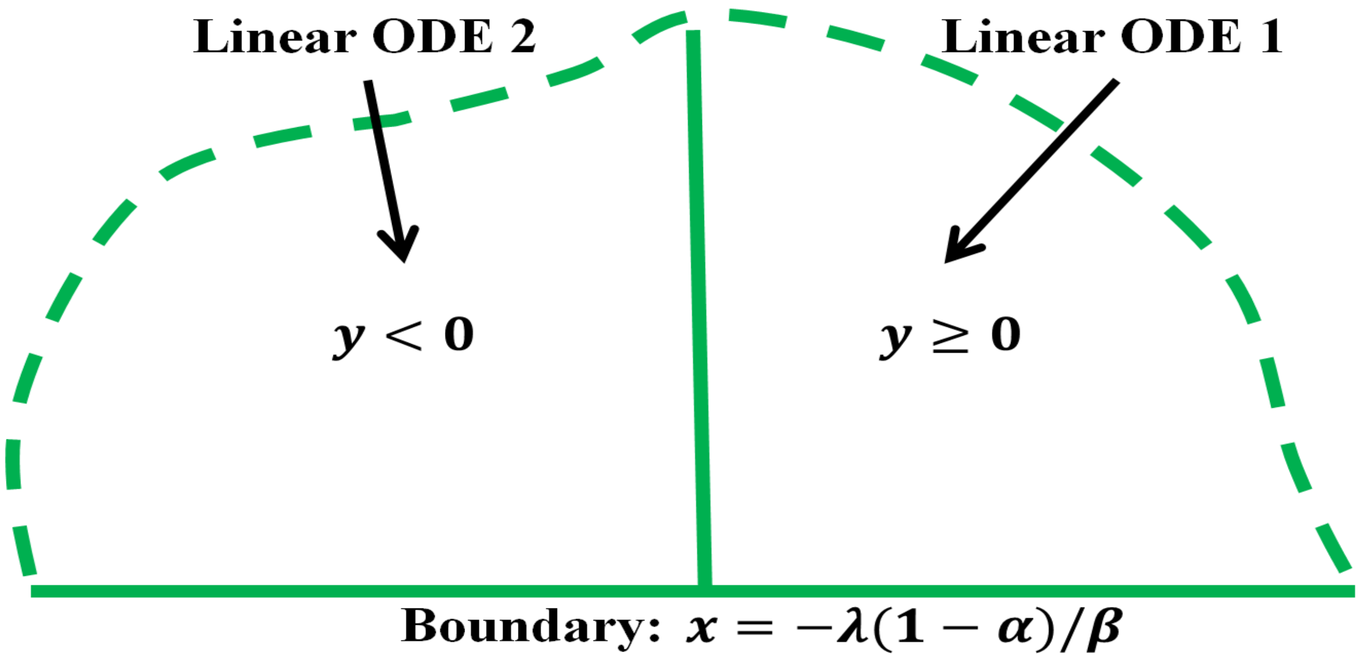
$(x, y, v)(t) \to (0,0,0)$, as $t \to \infty$

Behavior of fluid limit trajectories

- Fluid limit trajectories have complicated behavior

  - A “reflecting” boundary

  - Two domains where they follow different ODEs (but the RHS of the ODE is continuous everywhere)



Global vs. local stability

- Consider a dynamic system in $\mathbb{R}^3$ described by

$$\begin{cases} x' = -\gamma y' - \epsilon y \\ y' = \beta x + \alpha\mu(v - y^+) + \delta y^- - \theta y^+ \\ v' = \beta x - (1-\alpha)\mu(v - y^+) - \theta y^+ \end{cases} \tag{2}$$

Global vs. local stability

- Consider a dynamic system in $\mathbb{R}^3$ described by

$$\begin{cases} x' = -\gamma y' - \epsilon y \\ y' = \beta x + \alpha\mu(v - y^+) + \delta y^- - \theta y^+ \\ v' = \beta x - (1-\alpha)\mu(v - y^+) - \theta y^+ \end{cases} \qquad (2)$$

- Fluid limit is ***globally stable*** if every fluid limit trajectory converges to the equilibrium point (0,0,0).

- Fluid limit is ***locally stable*** if every solution of the dynamic system (2) converges to the equilibrium point (0,0,0).

Our main result (sufficient local stability conditions)

<u>Theorem 1</u>: Fluid limit is **locally stable** if either

$$\gamma > max\left\{\frac{\alpha\mu - \delta}{\beta}, \sqrt{\frac{(2-\alpha)\epsilon\mu + \alpha\epsilon\delta}{\beta\mu}}\right\} \qquad \text{(i)}$$

or

$$\gamma > max\left\{\frac{\alpha\mu - \delta + \sqrt{(\alpha\mu-\delta)^2 + 4\alpha\mu^2}}{2\beta}, \sqrt{max\left\{\frac{\alpha\epsilon(\delta-\mu)}{\beta\mu}, 0\right\}}\right\} \qquad \text{(ii)}$$

Some existing theory

- Even without boundary on X, we have ODE with **2 domains** ($y \geq 0$ and $y < 0$). (switched linear system in control theory)

- For local stability (stability of the system without boundary), **it is sufficient that Common Quadratic Lyapunov Function (CQLF) exists**.

- There is literature on existence of CQLF for switched linear systems. [R. Shorten et al., 2007; H. Lin, P. J. Antsaklis, 2006]

Fluid limit dynamics (when away from boundary)

- Fluid limit dynamics when away from boundary

$$\begin{cases} x' = -\gamma y' - \epsilon y \\ y' = \beta x + \alpha\mu(v - y^+) + \delta y^- - \theta y^+ \\ v' = \beta x - (1-\alpha)\mu(v - y^+) - \theta y^+ \end{cases} \quad (2)$$

- 2 domains:

$$y \geq 0 \qquad \begin{cases} x' = (-\gamma\beta)x + (\gamma\alpha\mu + \gamma\theta - \epsilon)y + (-\gamma\alpha\mu)v \\ y' = (\beta)x + (-\alpha\mu - \theta)y + (\alpha\mu)v \\ v' = (\beta)x + \big((1-\alpha)\mu - \theta\big)y + (-(1-\alpha)\mu)v \end{cases}$$

$$y < 0 \qquad \begin{cases} x' = (-\gamma\beta)x + (\gamma\delta - \epsilon)y + (-\gamma\alpha\mu)v \\ y' = (\beta)x + (-\delta)y + (\alpha\mu)v \\ v' = (\beta)x + (-(1-\alpha)\mu)v \end{cases}$$

Fluid limit dynamics (when away from boundary)

- In matrix form: $u(t) = (x(t), y(t), v(t))^T$

$$y \geq 0$$
$$u'(t) = A_1 u(t)$$

$$A_1 = \begin{pmatrix} -\gamma\beta & \gamma\alpha\mu + \gamma\theta - \epsilon & -\gamma\alpha\mu \\ \beta & -\alpha\mu - \theta & \alpha\mu \\ \beta & -(1-\alpha)\mu & -(1-\alpha)\mu \end{pmatrix}$$

$$y < 0$$
$$u'(t) = A_2 u(t)$$

$$A_2 = \begin{pmatrix} -\gamma\beta & \gamma\delta - \epsilon & -\gamma\alpha\mu \\ \beta & -\delta & \alpha\mu \\ \beta & 0 & -(1-\alpha)\mu \end{pmatrix}$$

Existence of CQLF

Necessary and sufficient condition for the existence of CQLF for switched linear systems [R. Shorten et al, 2007]

Proposition 1: Let $\boldsymbol{A_1}$ **and** $\boldsymbol{A_2}$ **be Hurwitz matrices** in $\mathbb{R}^{n\times n}$, where the difference $\boldsymbol{A_1 - A_2}$ **has rank one**. Then the two systems

$$u'(t) = A_1 u(t) \quad \text{and} \quad u'(t) = A_2 u(t)$$

have a CQLF if and only if the matrix product $\boldsymbol{A_1 A_2}$ **has no negative real eigenvalues**.

Theorem 1: Proof outline

- A_1 is always Hurwitz
- A_2 is Hurwitz if $\gamma > \frac{\alpha\mu - \delta}{\beta}$
- $rank(A_1 - A_2) = 1$

Theorem 1: Proof outline

- A_1 is always Hurwitz
- A_2 is Hurwitz if $\gamma > \frac{\alpha\mu-\delta}{\beta}$
- $rank(A_1 - A_2) = 1$
- KEY PART: $A_1 A_2$ has no negative real eigenvalues if either (i) or (ii) holds

$$\gamma > max\left\{\frac{\alpha\mu-\delta}{\beta}, \sqrt{\frac{(2-\alpha)\epsilon\mu+\alpha\epsilon\delta}{\beta\mu}}\right\} \qquad \text{(i)}$$

$$\gamma > max\left\{\frac{\alpha\mu-\delta+\sqrt{(\alpha\mu-\delta)^2+4\alpha\mu^2}}{2\beta}, \sqrt{max\left\{\frac{\alpha\epsilon(\delta-\mu)}{\beta\mu}, 0\right\}}\right\} \qquad \text{(ii)}$$

Theorem 1: Proof outline

Proposition 2 [R. Shorten et al, 2004]: If ${A_1}^{-1}$ is non-singular, the product $A_1 A_2$ has no negative eigenvalues if and only if ${A_1}^{-1} + \tau A_2$ is non-singular for all $\tau \geq 0$.

Theorem 1: Proof outline

Proposition 2 [R. Shorten et al, 2004]: If ${A_1}^{-1}$ is non-singular, the product $A_1 A_2$ has no negative eigenvalues if and only if ${A_1}^{-1} + \tau A_2$ is non-singular for all $\tau \geq 0$.

- $\det\left[{A_1}^{-1} + \tau A_2\right] < 0$ if either (i) or (ii) holds

$$\gamma > max\left\{\frac{\alpha\mu - \delta}{\beta}, \sqrt{\frac{(2-\alpha)\epsilon\mu + \alpha\epsilon\delta}{\beta\mu}}\right\} \qquad \text{(i)}$$

$$\gamma > max\left\{\frac{\alpha\mu - \delta + \sqrt{(\alpha\mu-\delta)^2 + 4\alpha\mu^2}}{2\beta}, \sqrt{max\left\{\frac{\alpha\epsilon(\delta-\mu)}{\beta\mu}, 0\right\}}\right\} \qquad \text{(ii)}$$

Some useful corollaries (sufficient local stability conditions)

Corollary 1: Given all other parameters are fixed, fluid limit is locally stable for all sufficiently large γ

Corollary 2: If $\alpha\mu \leq \delta$, then fluid limit is locally stable for all sufficiently small ϵ

Corollary 3: If $\alpha\mu > \delta$ and $\epsilon \leq \frac{(\alpha\mu-\delta)^2\mu}{(2-\alpha)\mu\beta+\alpha\delta\beta}$, then fluid limit is locally stable under condition

$$\gamma > \frac{\alpha\mu - \delta}{\beta}$$

Corollary 4: If $\mu > \delta$, then fluid limit is locally stable under condition

$$\gamma > \frac{\alpha\mu - \delta + \sqrt{(\alpha\mu - \delta)^2 + 4\alpha\mu^2}}{2\beta}$$ (does not depend on ϵ)

Corollary 5: If $\alpha = 0$, then fluid limit is locally stable for all positive $\beta, \mu, \epsilon, \gamma,$ and $\delta \geq 0, \theta \geq 0$

Numerical and simulation results

- We simulate the true system, with the boundary.
- We vary parameters and initial conditions.

Numerical and simulation results

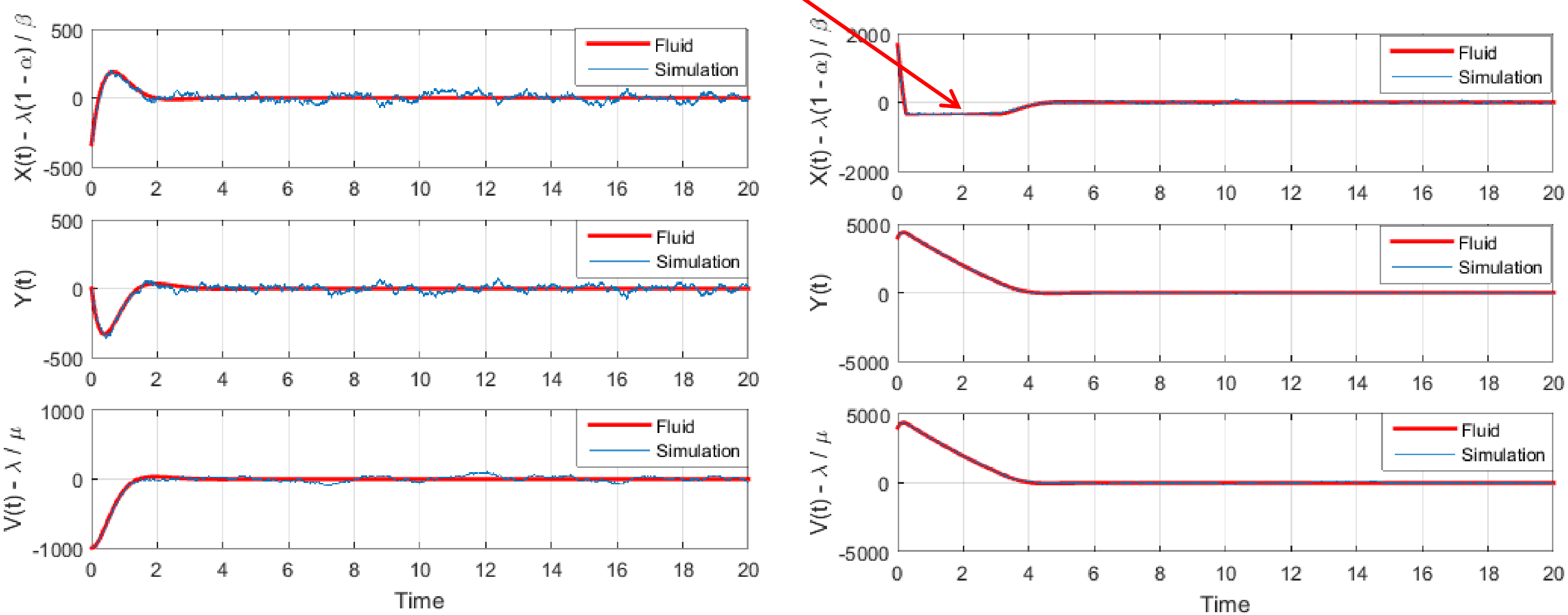
- We simulate the true system, with the boundary.
- We vary parameters and initial conditions.
- **Is there a gap between local and global stability?**

Numerical and simulation examples: Example 1

- The sufficient local stability conditions are satisfied

$$\Lambda = 2000\,, \alpha = 0.5\,, \beta = 3\,, \mu = 2\,, \gamma = 1\,, \epsilon = 1.5\,, \delta = 1\,, \theta = 0.1$$

Trajectory hits boundary on *x*



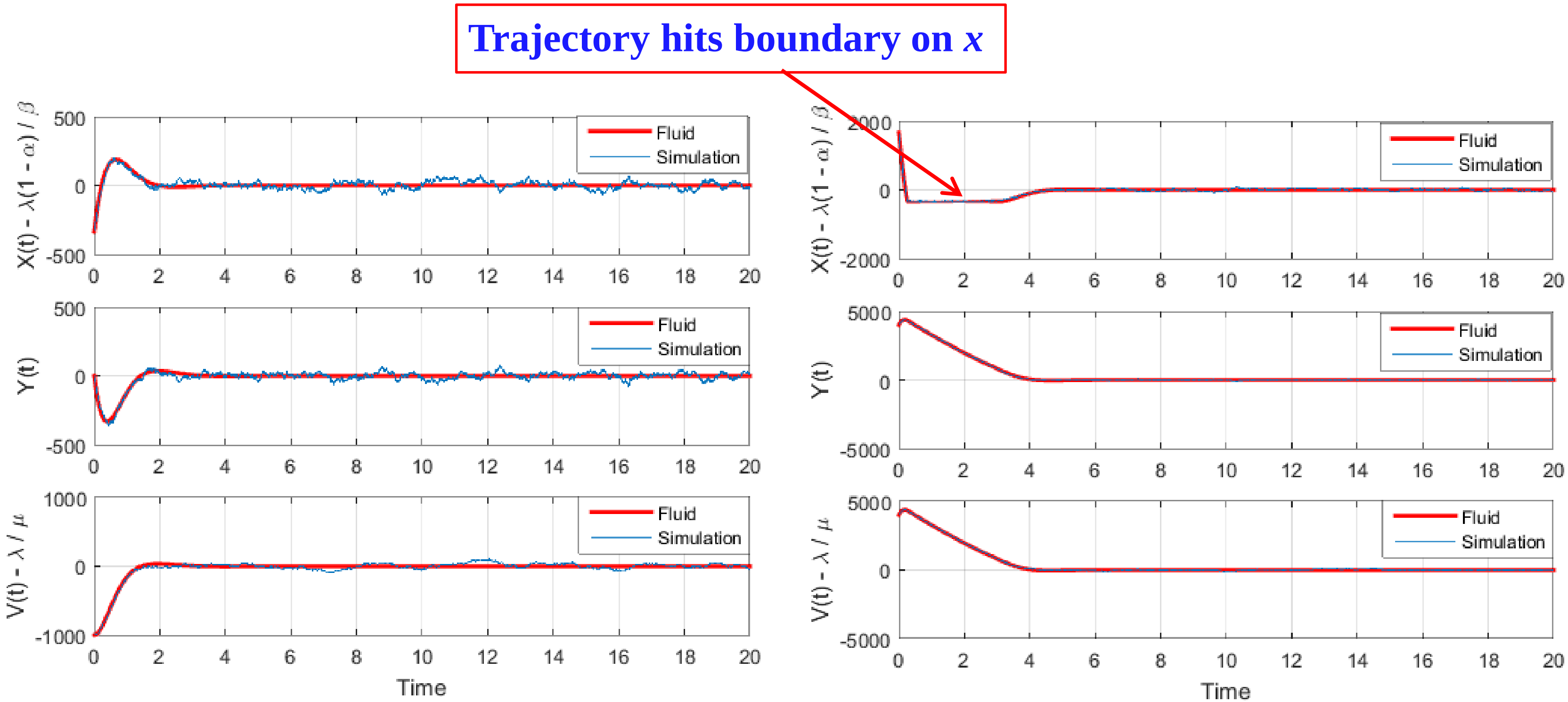
$\big(X(0), Y(0), Z(0)\big) = (0{,}0{,}0)$ $\big(X(0), Y(0), Z(0)\big) = (2000{,}4000{,}1000)$

Different initial conditions

Numerical and simulation examples: Example 1

- The sufficient local stability conditions are satisfied

$$\Lambda = 2000\,, \alpha = 0.5\,, \beta = 3\,, \mu = 2\,, \gamma = 1\,, \epsilon = 1.5\,, \delta = 1\,, \theta = 0.1$$



$\big(X(0), Y(0), Z(0)\big) = (0,0,0)$ $\big(X(0), Y(0), Z(0)\big) = (2000,4000,1000)$

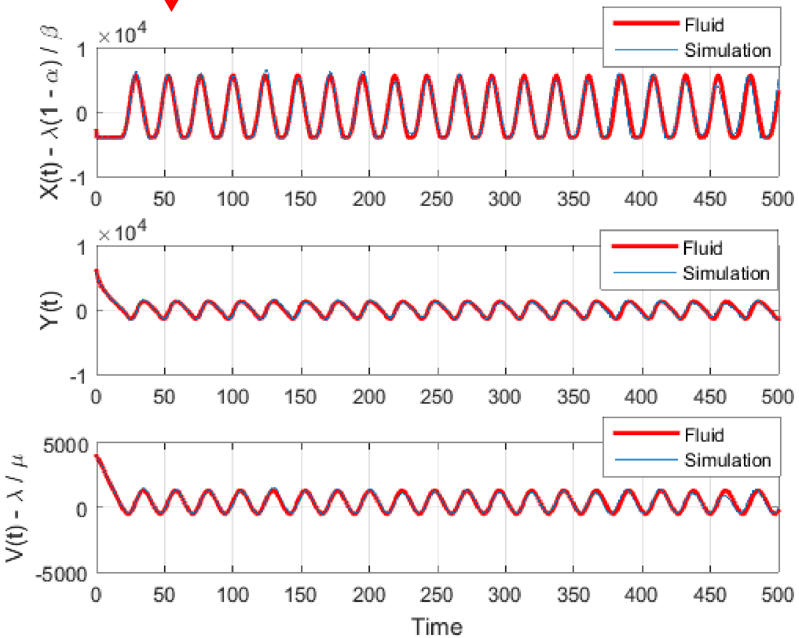
Different initial conditions

<u>Conjecture 1</u>: Our system is globally stable if it is locally stable.

Numerical and simulation examples: Example 2

- Consider

$$\Lambda = 2000\ ,\alpha = 0.9\ ,\beta = 0.05\ ,\mu = 0.5\ ,\boldsymbol{\gamma} = \mathbf{1}\ ,\epsilon = 1\ ,\delta = 0.01\ ,\theta = 0.01$$



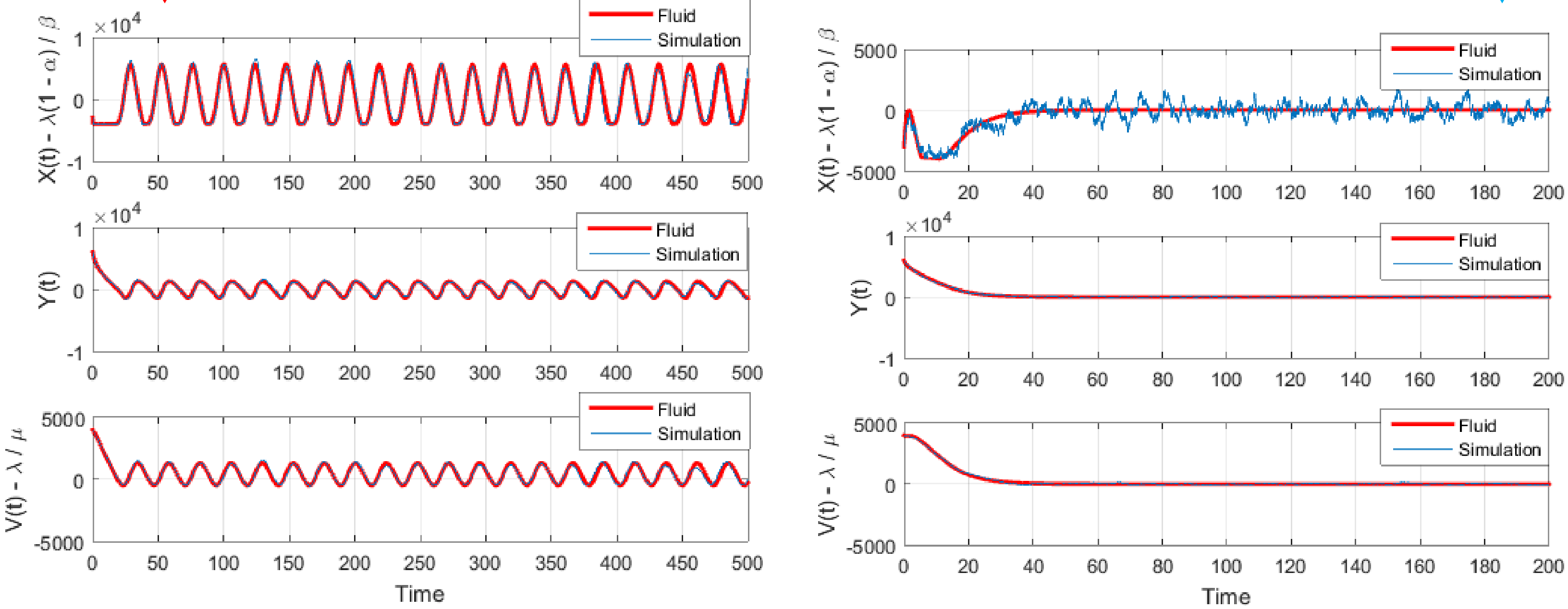
$$\big(X(0), Y(0), Z(0)\big) = (1000{,}6000{,}2000)$$

Numerical and simulation examples: Example 2

- Consider

$\Lambda = 2000\,, \alpha = 0.9\,, \beta = 0.05\,, \mu = 0.5\,, \boldsymbol{\gamma} = \mathbf{1}\,, \epsilon = 1\,, \delta = 0.01\,, \theta = 0.01$

$\Lambda = 2000\,, \alpha = 0.9\,, \beta = 0.05\,, \mu = 0.5\,, \boldsymbol{\gamma} = \mathbf{10}\,, \epsilon = 1\,, \delta = 0.01\,, \theta = 0.01$



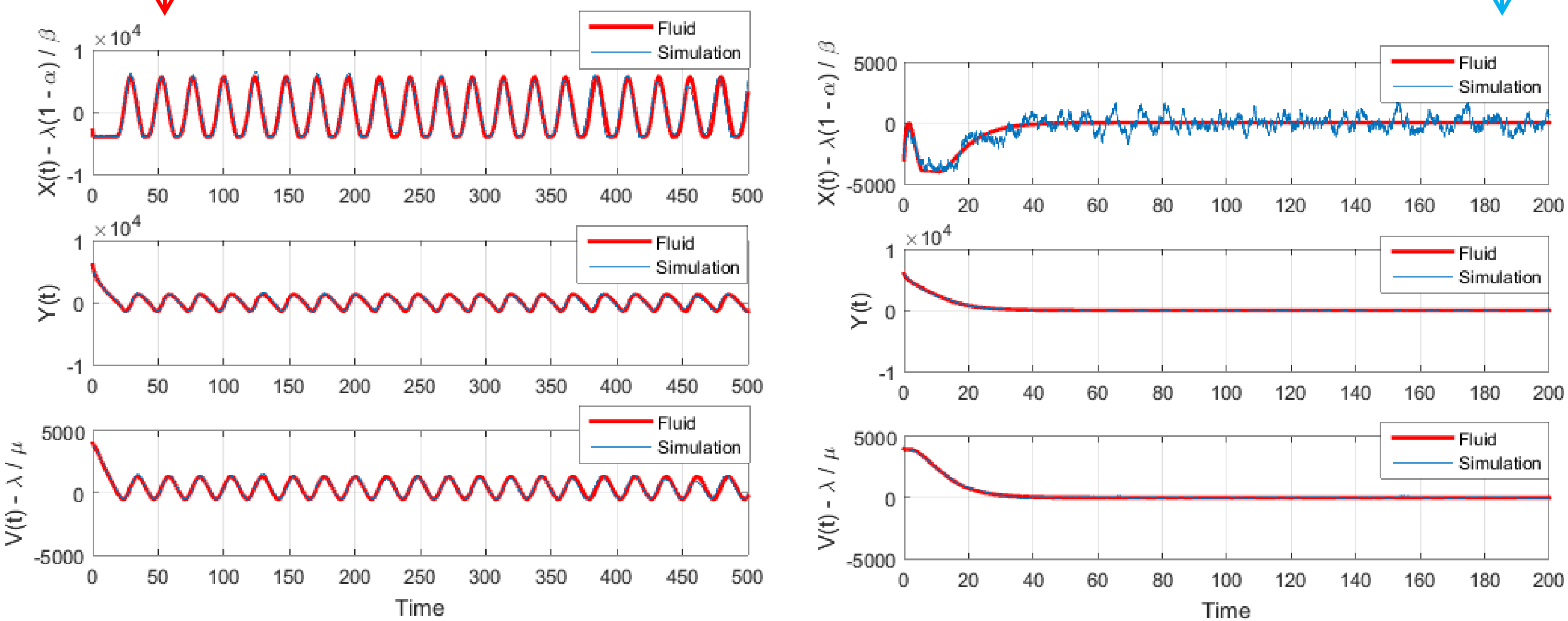
$(X(0), Y(0), Z(0)) = (1000{,}6000{,}2000)$

Numerical and simulation examples: Example 2

- Consider

$\Lambda = 2000\,, \alpha = 0.9\,, \beta = 0.05\,, \mu = 0.5\,, \boldsymbol{\gamma = 1}\,, \epsilon = 1\,, \delta = 0.01\,, \theta = 0.01$

$\Lambda = 2000\,, \alpha = 0.9\,, \beta = 0.05\,, \mu = 0.5\,, \boldsymbol{\gamma = 10}\,, \epsilon = 1\,, \delta = 0.01\,, \theta = 0.01$



$(X(0), Y(0), Z(0)) = (1000{,}6000{,}2000)$

Increasing γ from 1 to 10 makes system locally stable (Corollary 3). Simulation results indicate that it also makes fluid limit globally stable => supports our Conjecture 1.

Discussion and future work

- Global stability of fluid limit (including boundary behavior) – challenging
- Possible extensions: multi-class-customer system, multi-type-agent system, …
- Many applications and potential applications

References

- I. Gurvich, A. Ward. On the dynamic control of matching queues. *Stochastic Systems*, 4(2): 479-523, 2014.
- B. R. K. Kashyap. The double-ended queue with bulk service and limited waiting space. *Operations Research*, 14(5): 822-834, 1966.
- H. Lin, P. J. Antsaklis. Stability and stabilizability of switched linear systems: A survey of recent results. *IEEE Transactions on Automatic Control*, 54(2):308-322, 2009.
- L. Nguyen, A. L. Stolyar. A service system with randomly behaving on-demand agents. *SIGMETRICS Perform. Eval. Rev.*, 44(1), 365-366, 2016.
- G. Pang, A. L. Stolyar. A service system with on-demand agent invitations. *Queueing Systems*, 82(3), 259-283, 2016.
- R. Shorten, O. Mason, F. O'Cairbre, P. Curran. A unifying framework for the siso circle criterion and other quadratic stability criteria. *International Journal of Control*, 77(1): 1-9, 2004.
- R. Shorten, F. Wirth, O. Mason, K. Wulff, C. King. Stability criteria for switched and hybrid systems. *Society for Industrial and Applied Mathematics*, 49(4):545-592, 2007.
- A. L. Stolyar., M. I. Reiman, N. Korolev, V. Mezhibovsky, H. Ristock. Pacing in knowledge worker engagement. *United States Patent Application 20100266116-A1*, 2010.

THANK YOU !!!