A Service System with Randomly Behaving On-demand Agents

Lam Nguyen, Alexander Stolyar

Department of Industrial and Systems Engineering, Lehigh University

Introduction

Model: Agents are invited on-demand and join the system after a delay; may leave or return to the system after service completions.

Motivation: Call/contact centers, telemedicine, crowdsourcing-based customer service

Objective: Keep both customer and agent waiting times small

Results: Queue-length-based feedback scheme. Sufficient conditions for the local stability of fluid limits at the desired equilibrium point (with zero queues)

This model is a generalization of that in [2]. The details of our model, results, proofs, conjectures and numerical/simulation experiments are in [1].

Model

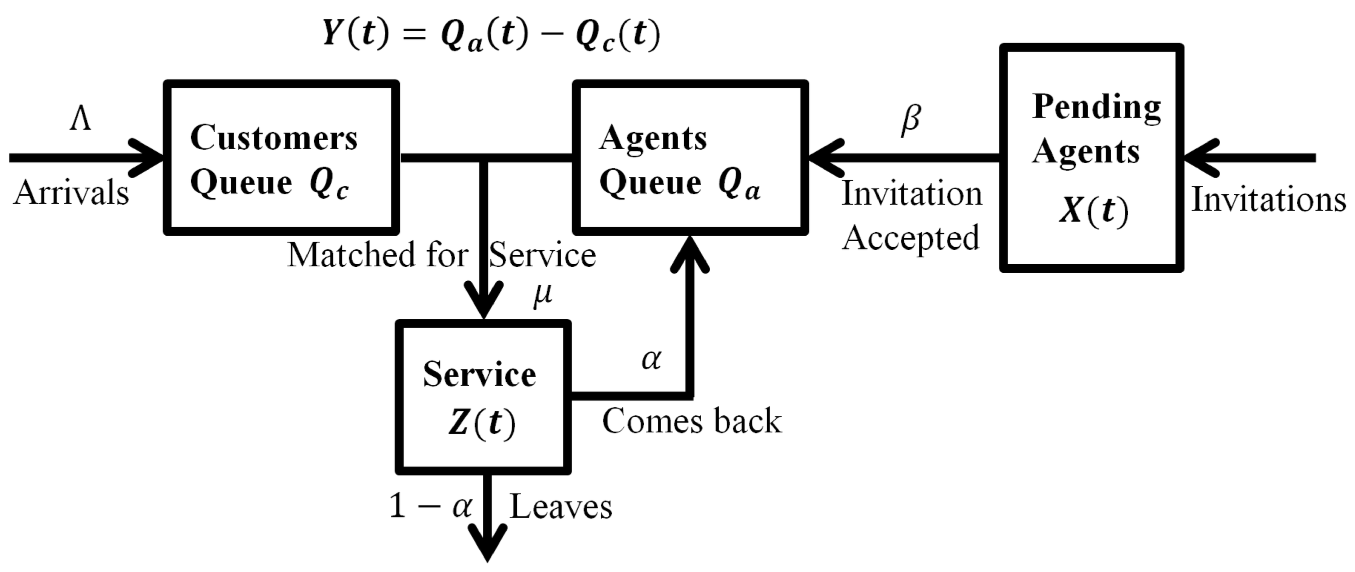


Figure 1: An Agent Invitation System

System state: $(X(t), Y(t), Z(t))$
System parameters: $\alpha \in (0,1)$, $\beta > 0$, $\mu > 0$

Algorithm

Algorithm parameters: $\gamma > 0$, $\epsilon > 0$

The algorithm controls the number of invited (pending) agents $X(t)$, which responds to different events during time dt as follows:

- **A customer arrival** with probability Λdt
$$\Delta X(t) = \gamma$$
- **An agent acceptance** with probability $\beta X(t) dt$
$$\Delta X(t) = -(\gamma \wedge X(t))$$
- **An additional event** with probability $\epsilon |Y(t)| dt$
$$\begin{aligned} &\Delta X(t) = -\text{sgn}(Y(t)) && \text{, if } X(t) \geq 1 \\ &\Delta X(t) = 1 && \text{, if } X(t) = 0 \text{ and } Y(t) < 0 \\ &\Delta X(t) = 0 && \text{, if } X(t) = 0 \text{ and } Y(t) \geq 0 \end{aligned}$$
- **A service completion** with probability $\mu Z(t) dt$
  - **Agent returns the agent queue** with probability α
$$\Delta X(t) = -(\gamma \wedge X(t))$$
  - **Agent leaves the system** with probability $1-\alpha$
$$\Delta X(t) = 0$$

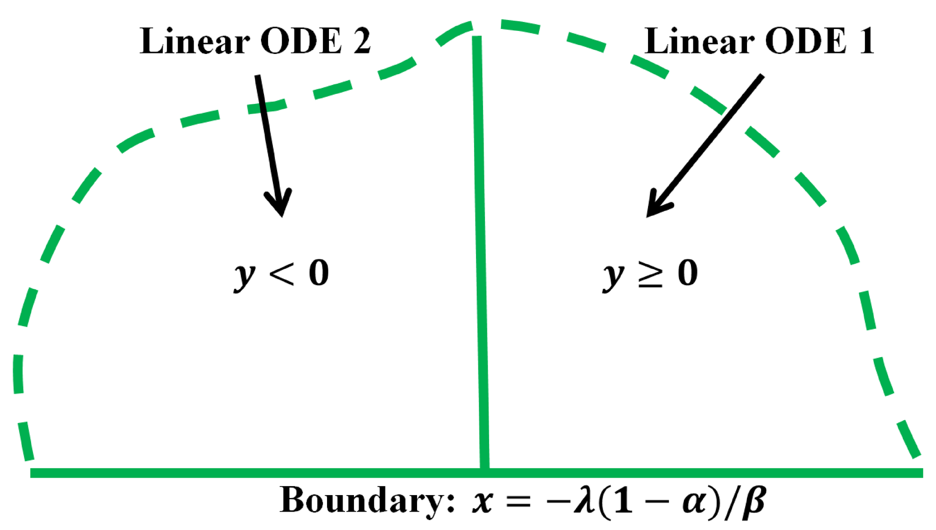
Fluid limit

Consider a sequence of systems, indexed by a scaling parameter $r \to \infty$. In the system with index r, the arrival rate is $\Lambda = \lambda r$, while the parameters α, β, μ, ϵ, γ are constant. The corresponding process is (X^r, Y^r, Z^r). Let $W = |Y| + 2Z$. Consider a new process (X^r, Y^r, W^r). Define fluid-scaled processes
$$(\bar{X}^r, \bar{Y}^r, \bar{W}^r) = r^{-1}(X^r - \lambda r(1-\alpha)/\beta, Y^r, W^r - 2\lambda r/\mu)$$

Fluid limit
$$(x(\cdot), y(\cdot), w(\cdot)) = \lim_{r \to \infty} (\bar{X}^r(\cdot), \bar{Y}^r(\cdot), \bar{W}^r(\cdot)) \qquad (1)$$
satisfies conditions
$$\begin{cases} x' = \begin{cases} -\gamma y' - \epsilon y, & \text{if } x > -\frac{\lambda(1-\alpha)}{\beta} \\ [-\gamma y' - \epsilon y] \vee 0, & \text{if } x = -\frac{\lambda(1-\alpha)}{\beta} \end{cases} \\ y' = \beta x + \frac{1}{2}\alpha\mu(w - |y|) \\ w' = \beta x + \frac{1}{2}(\alpha - 2)\mu(w - |y|) \end{cases} \qquad (2)$$

Fluid limit dynamics ignoring boundary on x

Consider a dynamic system in $\mathbb{R}^3$ described by ODE
$$\begin{cases} x' = -\gamma y' - \epsilon y \\ y' = \beta x + \frac{1}{2}\alpha\mu(w - |y|) \\ w' = \beta x + \frac{1}{2}(\alpha - 2)\mu(w - |y|) \end{cases} \qquad (3)$$
[(3) is (2) "away from boundary," i.e. when $x > -\frac{\lambda(1-\alpha)}{\beta}$]



Rewrite the system (3) as two linear systems $u'(t) = A^+ u(t)$ and $u'(t) = A^- u(t)$, where $u(t) = (x(t), y(t), w(t))^T$ and
$$y \geq 0: \quad A^+ = \begin{pmatrix} -\gamma\beta & \frac{1}{2}\gamma\alpha\mu - \epsilon & -\frac{1}{2}\gamma\alpha\mu \\ \beta & -\frac{1}{2}\alpha\mu & \frac{1}{2}\alpha\mu \\ \beta & -\frac{1}{2}(\alpha-2)\mu & \frac{1}{2}(\alpha-2)\mu \end{pmatrix}$$
and
$$y < 0: \quad A^- = \begin{pmatrix} -\gamma\beta & -\frac{1}{2}\gamma\alpha\mu - \epsilon & -\frac{1}{2}\gamma\alpha\mu \\ \beta & \frac{1}{2}\alpha\mu & \frac{1}{2}\alpha\mu \\ \beta & \frac{1}{2}(\alpha-2)\mu & \frac{1}{2}(\alpha-2)\mu \end{pmatrix}$$

We use the machinery of switched linear systems and common quadratic Lyapunov functions [3] to derive the following results

Main result: Local stability conditions

For any set of positive β, μ, and $\alpha \in (0,1)$, there exist values of $\gamma > 0$ and $\epsilon > 0$ satisfying either
$$\begin{cases} \epsilon < \frac{\beta\gamma^2}{2} - \frac{\alpha\gamma\mu}{2} \\ \gamma > \frac{\alpha\mu}{\beta} \end{cases} \quad \text{or} \quad \begin{cases} \frac{\beta\gamma^2}{4} < \epsilon < \frac{\beta\gamma^2}{2} \\ \epsilon > \frac{\beta\gamma^2}{2} - \left(\frac{\alpha\gamma\mu}{2} - \frac{(1-\alpha)\mu^2}{2\beta}\right) \\ \gamma > \frac{(1-\alpha)\mu}{\alpha\beta} \end{cases} \qquad (4)$$
For the parameters, satisfying either the left or right condition of (4), a common quadratic Lyapunov function of the system (3) exists, and the **system (3) is exponentially stable**.

Note that the **left condition of (4) is very easy to achieve in practice**. Indeed, for any given $\epsilon > 0$, it holds for all sufficiently large γ.

Global vs. local stability

Fluid limit is *globally stable* if every fluid limit trajectory converges to the equilibrium point $(0,0,0)$.

Fluid limit is *locally stable* if every trajectory of the dynamic system (3) converges to the equilibrium point $(0,0,0)$.

Numerical and simulation experiments

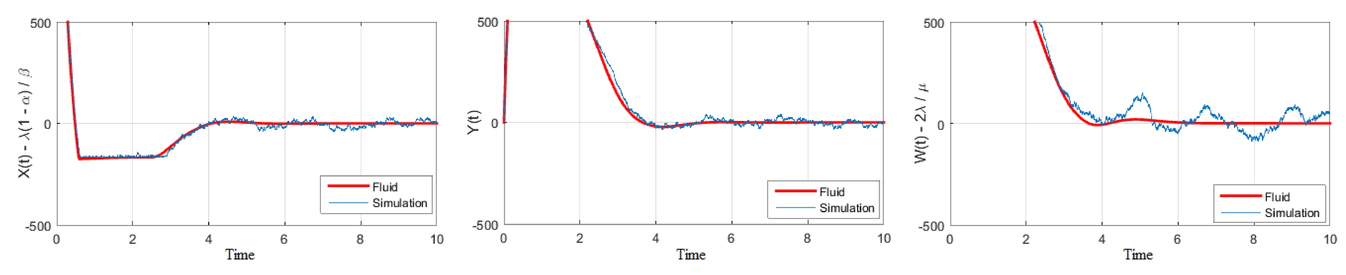


Figure 2: Local stability condition holds

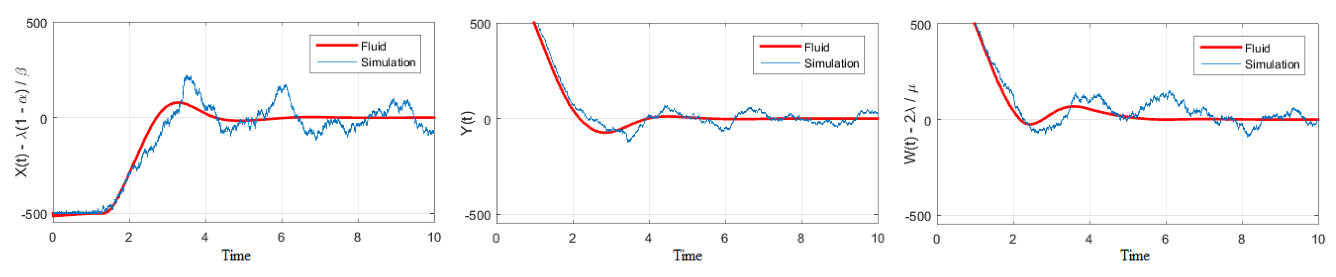


Figure 3: Local stability condition does not hold, but A^- is Hurwitz

Conjectures

Conjecture 1: For our system, fluid limit is globally stable if it is locally stable.

Conjecture 2: For our system, matrix A^- being Hurwitz is sufficient for local stability of fluid limit. (A^+ is always Hurwitz in our case.)

References

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[2] G. Pang and A. Stolyar.
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[3] R. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King.
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Contact Information

Lam M. Nguyen
http://coral.ise.lehigh.edu/lmn214/
lmn214@lehigh.edu

Alexander L. Stolyar
http://www.lehigh.edu/∼als714/
stolyar@lehigh.edu